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Free Energy and Critical Temperature in Eleven Dimensions

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Abstract

We compute the one-loop contribution to the free energy in eleven-dimensional supergravity, with the eleventh dimension compactified on a circle of radius R_{11}. We find a finite result, which, in a small radius expansion, has the form of the type IIA supergravity free energy plus non-perturbative corrections in the string coupling g_A, whose coefficients we determine. We then study type IIA superstring theory at finite temperature in the strong coupling regime by considering M-theory on $R^9 \times T^2$, one of the sides of the torus being the euclidean time direction, where fermions obey antiperiodic boundary conditions. We find that a certain winding membrane state becomes tachyonic above some critical temperature, which depends on g_A. At weak coupling, it coincides with the Hagedorn temperature, at large coupling it becomes $T_{\rm cr} \cong 0.31\ l_P^{-1}$ (so it is very small in string units).

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1. Introduction

A problem of interest in type IIA superstring theory is to understand the evolution of the degrees of freedom of the system as the coupling is increased from weak to strong values. For weak couplings, the theory can be described in terms of a supersymmetric relativistic string, but for strong coupling the relevant degrees of freedom are not well understood. The study of a system at finite temperature can give some non-trivial information about its microscopic degrees of freedom, and about their behavior as the system is heated up to high temperatures. In this paper we will discuss some features of M-theory at finite temperature, with the eleventh dimension X^{11} compactified on a circle of radius R_{11}. Given our limited knowledge of M-theory, a complete treatment is of course presently impossible. Nevertheless, we will find, somewhat surprisingly, that very interesting aspects can be revealed by simple calculations.

For small radius R_{11}, one must recover the thermodynamics of string theory, which is well understood qualitatively [1]. There is a first order phase transition at some $T_0 < T_H$, where T_H is the Hagedorn temperature [2], with a large latent heat leading to a gravitational instability of the thermal ensemble. The way the Hagedorn temperature is calculated in string theory basically involves free string theory methods, so an important question is how interactions modify this picture. In particular, one would like to understand what happens to the Hagedorn temperature in the type IIA superstring theory at strong coupling, in other words, what is the fate of the Hagedorn transition in $D = 11$.

The presence of phase transitions is usually reflected as infrared divergences in the one-loop free energy. This calculation is difficult to generalize to eleven dimensions, for a number of reasons. In string theory, the one-loop free energy is essentially the sum over free energy contributions of each individual physical string mode. In order to compute a one-loop free energy in M-theory in this way, a more detailed knowledge of the relevant physical degrees of freedom would be required. In addition, in the eleven dimensional theory there is no coupling constant parameter, and higher loops will give contributions to the free energy of order one. Nevertheless, as a first step, one could try to determine the free energy in eleven-dimensional supergravity. As a physical application, the eleven-dimensional supergravity result may be then used to incorporate non-perturbative D0 brane contributions to the one-loop free energy of string theory. $D = 11$ supergravity is not renormalizable as a quantum field theory. However, as we show in section 2, the one-loop free energy is finite, thanks to a cancellation between boson and fermion contributions (a general discussion of one-loop divergences in supergravity can be found in [3]). This is important, since the presence of an UV divergence (as happens in a purely bosonic theory), would prevent us from recovering the ten-dimensional physics in the small radius limit.

One of the difficulties in understanding details of the Hagedorn transition in string theory is that gravitational effects cannot be neglected shortly above the Hagedorn temperature, due to a large genus zero contribution to the free energy [1]. The description in

terms of a free string gas in a flat background is not applicable; rather, one expects gravitational collapse near the Hagedorn temperature. One can give a qualitative description of what physical picture should be expected (e.g. by using the microcanonical ensemble one can argue which string configurations are favored and dominate the density of states), but the arguments are mostly based on string propagation in flat space. A more detailed understanding was recently achieved in non-commutative open string theory, which does not contain gravitation [4].

In the case of M theory at large radius R_{11}, a flat-theory description of the thermal ensemble can be justified only for temperatures $T \ll O(l_P^{-1})$, where l_P is the eleven-dimensional Planck length. The reason is the following one. Statistical mechanics is valid provided the system has a large volume, so that it contains many degrees of freedom. In eleven dimensions, large volume means that the size R of the system is $R \gg l_P$. On the other hand, a flat theory description requires that corrections to the flat Minkowski metric are small, i.e. $\frac{G_N E}{R^8} \ll 1$, with $G_N \sim l_P^9$. Now consider the thermal ensemble at temperature $T = O(l_P^{-1})$. Then the energy density is $\frac{E}{R^{10}} = \text{const.} l_P^{-11}$, since there is no other parameter in the theory. Hence gravitational effects are of order $\frac{G_N E}{R^8} \sim \frac{R^2}{l_P^2}$, so they are important for $R > l_P$. Thus it is not possible to have a statistical description of the thermal ensemble in flat space near the Planck temperature. A temperature $T = O(l_P^{-1})$ is in fact the maximum temperature that a statistical system can reach in eleven dimensions without gravitational collapse. For any $T > O(l_P^{-1})$, a thermal ensemble with size $R > l_P$ will be inside its Schwarzschild radius, so it will collapse into a black hole. This can be compared with the situation in string theory, where one has the string coupling g_s as a free parameter, and for sufficiently small g_s gravity can be ignored at any $T < T_H$ (but not at $T > T_H$ due to the appearance of a genus zero contribution $F_0 \sim -1/g_s^2$ [1]).

Despite these complications, using flat-space methods to study the Hagedorn transition has led to important insights on the nature of string theory and its physical degrees of freedom. One may then expect that a similar simplified study in eleven dimensions can teach us important lessons about M-theory. Here we will find that the Hagedorn temperature admits a straightforward generalization to eleven dimensions. In string theory, the Hagedorn temperature can be found as the temperature at which a certain winding state becomes tachyonic [5] (see sect. 3). In eleven dimensions, this winding state is a winding membrane. We find that it becomes tachyonic at some critical temperature $T_{\rm cr} = T_{\rm cr}(g_A)$, where g_A is the type IIA string coupling, which smoothly interpolates between the Hagedorn temperature ($g_A \ll 1$) and a critical temperature $T = O(l_P^{-1})$ ($g_A \gg 1$). This is done in sect. 4, where we also include some remarks about a duality to type 0A string theory [6].

Other discussions about the Hagedorn temperature in string theory can be found e.g. in [7-10]. There have also been some discussions on membrane theory at finite temperature in [11], where there is an attempt of computing the free energy. The matrix theory approach [12] has also been considered at finite temperature in refs. [13-16], which discuss other aspects of the Hagedorn transition. There is no overlap with the present work.

In appendix A we review standard properties of the non-holomorphic Eisenstein series [17]. Appendix B contains new Eisenstein-type series with alternating signs in the sum. We derive formulas for the expansions at large and small values of the modular parameter. In the main text, these series arise as the contribution of antiperiodic fermions.

2. Free energy in eleven dimensions

The thermal ensemble at temperature T can be studied as usual by considering the theory in Euclidean space where the time coordinate is compactified on a circle of circumference $1/T$, i.e.

$$X^0 = X^0 + 2\pi R_0 \ , \qquad T = (2\pi R_0)^{-1} \ , \tag{2.1}$$

where fermions obey antiperiodic boundary conditions. In order to have a description of type IIA superstring theory which can be extended beyond perturbation theory, one should thus consider euclidean M-theory on a 2-torus X^0, X^{11},

$$X^0 = X^0 + 2\pi R_0 \ , \qquad X^{11} = X^{11} + 2\pi R_{11} \ .$$

where fermions are antiperiodic around X^0, and periodic around X^{11}. Here we will consider the case of a rectangular torus.

The supersymmetric compactification of M-theory on a 2-torus gives rise to a theory that inherits the $SL(2,Z)$ isometry of the 2-torus, because this symmetry is not broken by boundary conditions of the fields. In the case of finite temperature M-theory, the different boundary conditions for fermions in the directions X^0, X^{11} break the $SL(2,Z)$ symmetry. This will be reflected in the present calculation of the free energy. Instead, in a purely bosonic theory, the exact partition function must have the symmetry

$$Z_{\rm bos}(g_{\rm eff}) = Z_{\rm bos}(1/g_{\rm eff}) \ , \tag{2.2}$$

with

$$g_{\rm eff} \equiv \frac{R_{11}}{R_0} = 2\pi\sqrt{\alpha'}T g_A \ . \tag{2.3}$$

Here $g_A = R_{11}/\sqrt{\alpha'}$ is the type IIA string coupling. This implies for the free energy the relation:

$$F_{\rm bos}(g_{\rm eff}, A, l_P) = g_{\rm eff} F_{\rm bos}(\frac{1}{g_{\rm eff}}, A, l_P) \ , \tag{2.4}$$

where $A = R_0 R_{11}$. The symmetry (2.4) can also be expressed in terms of string theory parameters $\{g_A, T, \alpha'\}$ and relates low and high temperature regimes, as well as weak and strong coupling regimes.

Here we will obtain the one-loop contribution to the free energy in eleven-dimensional supergravity compactified on a circle by adding to the ten-dimensional supergravity expression an extra factor containing a sum over Kaluza-Klein modes $\sum_m e^{-\pi\tau m^2/R^2}$. The free energy in ten-dimensional supergravity can be obtained from superstring theory as a limit $\alpha' \to 0$ (deriving the one-loop contribution to the free energy directly from the component formulation of $D = 11$ supergravity is more complicated). This is similar to [18,19], where the one-loop 4-graviton amplitude in $D = 11$ supergravity was computed by adding the Kaluza-Klein modes to the $D = 10$ supergravity amplitude.

2.1. Free energy in a simplified model

Before considering the free energy in string theory, it is instructive to study a simplified model in which we sum up the individual free energies of Kaluza-Klein scalar fields. That is

$$F(T) = VT \sum_m \int \frac{d^{D-2}p}{(2\pi)^{D-2}} \log\left[1 - e^{-\omega_p/T}\right] , \quad \omega_p^2 = \vec{p}^2 + \frac{m^2}{R_{11}^2} . \tag{2.5}$$

Expanding the logarithm and using

$$e^{-2\sqrt{ab}} = \sqrt{\frac{b}{\pi}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-at-b/t} ,$$

one can write

$$F(T) = -V \sum_m \int_0^\infty \frac{dt}{t^{\frac{D+1}{2}}} \sum_{w=1}^\infty \exp\left[- \frac{\pi m^2 t}{R_{11}^2} - \frac{\pi w^2 R_0^2}{t}\right] , \quad R_0 = (2\pi T)^{-1} , \tag{2.6}$$

where we have ignored a multiplicative numerical constant. Let us now set $D = 11$ (so that $[V]$=cm^9). Including the vacuum part $w = 0$, and making a Poisson resummation in w, i.e.

$$\sum_{w=-\infty}^\infty e^{-\frac{\pi R^2}{t} w^2} = \frac{\sqrt{t}}{R} \sum_{k=-\infty}^\infty e^{-\pi t \frac{k^2}{R^2}} , \tag{2.7}$$

we get

$$F(T) = -\pi T V \int_0^\infty \frac{dt}{t^{11/2}} \sum_{k,m} \exp\left[- \pi t \left(\frac{k^2}{R_0^2} + \frac{m^2}{R_{11}^2}\right)\right] . \tag{2.8}$$

This can be recognized as the one loop contribution of Kaluza-Klein scalars associated with the 2-torus X^0, X^{11}. The integral diverges in the ultraviolet region $t \to 0$. We can isolate the divergent piece by performing Poisson resummation in k, m and introducing the new integration variable $s = 1/t$, so that

$$F(T) = -\tfrac{1}{2} V R_{11} \int_0^\infty \frac{ds}{s} s^{11/2} \sum_{w,w'} \exp\left[- \pi s \left(w^2 R_0^2 + {w'}^2 R_{11}^2\right)\right] \tag{2.9}$$

Now the UV divergence is in the term $(w, w') = 0$. In string theory the analog term will cancel against a fermion contribution. Thus we get

$$F(T) = -\frac{V R_{11} \Gamma(11/2)}{2\pi^{11/2}} \sum_{(w,w')\neq(0,0)} \left(w^2 R_0^2 + {w'}^2 R_{11}^2\right)^{-\frac{11}{2}} + \text{ divergent term} . \tag{2.10}$$

This can be expressed in terms of an Eisenstein series (see appendix A)

$$F(T) = -\frac{V R_{11} \Gamma(11/2)}{(\pi R_{11} R_0)^{\frac{11}{2}}} \zeta(11) E_{\frac{11}{2}}(g_{\text{eff}}) + \text{ divergent term} . \tag{2.11}$$

It satisfies the symmetry relation mentioned above, $F(g_{\rm eff},A) = g_{\rm eff} F(1/g_{\rm eff},A)$ (this symmetry still holds for the regularized divergent part for a cutoff proportional to l_P^{-1}, i.e. independent of the radii R_0, R_{11}). To study the behavior at $g_{\rm eff} \gg 1$ and $g_{\rm eff} \ll 1$ we use the expansions (A.7), (A.8). We obtain

$$\frac{F(T)}{V} = -\frac{945\zeta(11)}{32\pi^5 R_{11}^{10}} - \frac{24\zeta(10)}{\pi^5 R_0^{10}} + O\big(\exp[-2\pi\frac{R_0}{R_{11}}]\big) \ , \qquad R_{11} \ll R_0 \ , \tag{2.12}$$

$$\frac{F(T)}{V} = -\frac{945\zeta(11) R_{11}}{32\pi^5 R_0^{11}} - \frac{24\zeta(10)}{\pi^5 R_0 R_{11}^{9}} + O\big(\exp[-2\pi\frac{R_{11}}{R_0}]\big) \ , \qquad R_{11} \gg R_0 \ . \tag{2.13}$$

The leading term in eq. (2.13) has the correct form for the free energy of a massless field theory in $D = 11$. The expression (2.12) contains two terms with power-like dependence on R_{11}/R_0. The subleading term proportional to $1/R_0^{10}$ gives the expected expression for the free energy of a $D = 10$ massless field theory. However, there is a leading term proportional to $1/R_{11}^{10}$. The presence of a term of the form $1/R_{11}^{10}$ in the supergravity calculation at $R_{11} \ll R_0$ would be problematic because there is no such contribution in superstring theory at weak coupling. As we shall see below, in the supergravity calculation the analog term cancels out.

2.2. *One-loop free energy in $D = 11$ supergravity*

Let us now consider the supergravity computation. The calculation of the free energy in type II superstring theory was carried out in [1] in the genus one approximation (valid for $g_A \ll 1$), with the result

$$F_{\rm string} = -\frac{1}{4} V (4\pi^2\alpha')^{-5} \int_{\mathcal F} \frac{d^2\tau}{\tau_2^6}\ \big|\eta(\tau)\big|^{-24} \sum_{w',w} e^{-\frac{\pi r_0^2}{\tau_2}|w'+w\tau|^2}$$
$$\times \Big[(|\theta_2|^8 + |\theta_3|^8 + |\theta_4|^8)(0,\tau) + e^{i\pi(w+w')}(\theta_2^4\bar\theta_4^4 + \theta_4^4\bar\theta_2^4)(0,\tau)$$
$$- e^{i\pi w'}(\theta_2^4\bar\theta_3^4 + \theta_3^4\bar\theta_2^4)(0,\tau) - e^{i\pi w}(\theta_3^4\bar\theta_4^4 + \theta_4^4\bar\theta_3^4)(0,\tau)\Big] \ , \qquad r_0^2 \equiv \frac{R_0^2}{\alpha'} \ . \tag{2.14}$$

In order to obtain the ten-dimensional supergravity result, we first separate the term with vanishing winding $w = 0$, writing

$$F_{\rm string} = F'_{\rm string} + F_0$$

where $F'_{\rm string}$ is as in (2.14) with the omission of the $w = 0$ term in the sum, which we call F_0. The free energy in ten-dimensional supergravity is obtained by taking the limit

$\alpha' \to 0$ in F_0. Taking $\alpha' \to 0$ implies keeping the leading terms of the theta functions and Dedekind η function at large τ_2, that is

$$\eta(\tau) \cong q^{1/12}(1-q^2) \ , \quad \theta_2(0,\tau) \cong 2q^{1/4}(1+q^2) \ , \quad q \equiv e^{i\pi\tau} \ ,$$

$$\theta_3(0,\tau) \cong 1+2q+O(q^3) \ , \quad \theta_4(0,\tau) \cong 1-2q+O(q^3) \ .$$

We obtain

$$F_{\rm SG}^{(10)} = \lim_{\alpha'\to 0} F_0 = -256V(4\pi^2\alpha')^{-5} \int \frac{d\tau_2 d\tau_1}{\tau_2^6} \sum_{w'} \left[1-(-1)^{w'}\right] e^{-\frac{\pi r_0^2}{\tau_2} w'^2} \ , \tag{2.15}$$

where the integration region is now the whole strip $\tau_2 > 0$, $|\tau_1| < 1/2$.

Integrating over τ_1 and making a Poisson resummation in w', we get

$$F_{\rm SG}^{(10)} = -256 \frac{V}{r_0} (4\pi^2\alpha')^{-5} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \sum_k \left[e^{-\frac{\pi\tau_2}{r_0^2}k^2} - e^{-\frac{\pi\tau_2}{r_0^2}(k+\frac{1}{2})^2} \right] . \tag{2.16}$$

The appearance of half-integer momentum modes is a well-known distinctive feature of having fermions with antiperiodic boundary conditions. Similar expressions for the partition function or for the free energy appear in string compactifications where fermions obey antiperiodic boundary conditions around some spatial dimension [20] (for recent discussions on string compactifications with antiperiodic fermions, see e.g. [21,22]). Taking a similar limit on $F'_{\rm string}$, and making a Poisson resummation in w', gives

$$\frac{F'_{\rm string}}{VT} \to -(4\pi^2\alpha')^{-\frac{9}{2}} \int \frac{d\tau_2 d\tau_1}{\tau_2^{11/2}}\ e^{2\pi\tau_2} \sum_{k,w} \left[1-(-1)^w\right]\ e^{-\pi\tau_2(w^2 r_0^2 + \frac{k^2}{r_0^2})}\ e^{2\pi i \tau_1 k w} . \tag{2.17}$$

One can see the presence of the thermal tachyon corresponding to the term $k=0$, $w=\pm 1$, which reflects as an infrared divergence of $F_{\rm string}$ for $T > T_H$: the integral is divergent at $\tau_2 \to \infty$ for $r_0^2 < 2$ (this is precisely the critical radius that one obtains by examining the spectrum, see eq. (3.4) with $a_L = a_R = 1/2$).

The Kaluza-Klein modes associated with the eleventh dimension have masses $|m|/g_A$ in string units, m=integer. By adding their contribution to $F_{\rm SG}^{(10)}$, given in (2.16) , we get the one-loop contribution to the free energy in eleven-dimensional supergravity compactified on a circle X^{11}. This is

$$F_{\rm SG}^{(11)} = -256 \frac{V}{r_0} (4\pi^2\alpha')^{-5} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \sum_{k,m} e^{-\frac{\pi\tau_2}{g_A^2}m^2} \left[e^{-\frac{\pi\tau_2}{r_0^2}k^2} - e^{-\frac{\pi\tau_2}{r_0^2}(k+\frac{1}{2})^2} \right] . \tag{2.18}$$

The first term (containing $e^{-\frac{\pi\tau_2}{r_0^2}k^2}$) is essentially the same as the expression obtained in the previous bosonic example (2.8). The second term (with $e^{-\frac{\pi\tau_2}{r_0^2}(k+\frac{1}{2})^2}$) represents the fermion contribution. Making Poisson resummation in both k, m we get

$$F_{\rm SG}^{(11)} = -256 V g_A (4\pi^2\alpha')^{-5} \int_0^\infty \frac{ds}{s} s^{11/2} \sum_{w',n} \left[1-(-1)^{w'}\right] e^{-\pi s(w'^2 r_0^2 + n^2 g_A^2)} \ , \qquad (2.19)$$

i.e.

$$F_{\rm SG}^{(11)} = -256 V g_A (4\pi^2\alpha')^{-5} \frac{\Gamma(11/2)}{(\pi r_0 g_A)^{11/2}} \sum_{w',n} \left[1-(-1)^{w'}\right] \frac{g_{\rm eff}^{11/2}}{(w'^2+n^2 g_{\rm eff}^2)^{11/2}} \ , \qquad (2.20)$$

with $g_{\rm eff} = \frac{R_{11}}{R_0} = \frac{g_A}{r_0}$. This can be written in terms of the Eisenstein-type series defined in appendices A, B as follows:

$$\frac{F_{\rm SG}^{(11)}}{V} = -T^{10} \frac{2^9 \Gamma(11/2)\zeta(11)}{\pi^{11/2} g_{\rm eff}^{9/2}} \left[E_{\frac{11}{2}}(g_{\rm eff}) - F_{\frac{11}{2}}(g_{\rm eff})\right] \ . \qquad (2.21)$$

Using the formulas (A.7), (A.8), (B.11), (B.12) for the weak and strong coupling expansions, we obtain

$$\frac{F_{\rm SG}^{(11)}}{VT} = -\frac{24\zeta(10)}{\pi^5}(2^{10}-1)\, T^9 \ + \ O(e^{-2\pi/g_{\rm eff}}) \ , \quad g_{\rm eff} \ll 1 \ , \qquad (2.22)$$

and

$$\frac{F_{\rm SG}^{(11)}}{2\pi R_{11} VT} = -\frac{945\zeta(11)}{64\pi^5}(2^{11}-1)\, T^{10} - \frac{3\ 2^{12}\zeta(10)}{\pi^5} \frac{T^{10}}{g_{\rm eff}^{10}} + O(e^{-2\pi g_{\rm eff}}) \ , \quad g_{\rm eff} \gg 1 \ . \quad (2.23)$$

The weak coupling expression (2.22) has the expected field theory behavior $\frac{F}{VT} \sim T^{D-1}$ for a free energy of a $D=10$ dimensional massless field theory. The leading term is in fact $F_{\rm SG}^{(10)}$ given in (2.16). The strong $g_{\rm eff} \gg 1$ coupling expression (2.23) has the expected field theory behavior $\frac{F}{R_{11}VT} \sim T^{D-1}$ for a free energy of a $D=11$ dimensional massless field theory. This agrees with the expectation that varying g_A from small to large values should lead to an interpolation of a ten-dimensional and an eleven-dimensional theory. This does not happen in the bosonic theory, which has an extra term at small coupling (the underlying reason for which in the bosonic theory the small radius limit does not give the ten dimensional result is the UV divergence, which is different in ten and eleven dimensions; consequently, some memory of the KK modes survives even at small radius R_{11}). Note also that the only power-like correction in eq. (2.23) is always subleading, since

$g_{\rm eff} > 1$. It is independent of the temperature. Indeed, as a function of T, g_A, the free energy in (2.23) has the form

$$\frac{F_{\rm SG}^{(11)}}{2\pi R_{11} VT} = -\frac{945\zeta(11)}{64\pi^5}(2^{11}-1)\, T^{10} - \frac{12\zeta(10)}{\pi^{15}}\frac{1}{(\sqrt{\alpha'} g_A)^{10}} + O(e^{-2\pi g_{\rm eff}})\ , \quad g_{\rm eff} \gg 1. \tag{2.24}$$

Finally, one can use the above results to define an improved expression for the free energy of type IIA superstring theory by adding to the one-loop expression (2.14) (representing the contribution of perturbative string modes) the contribution of D0 branes represented by the exponentially small terms in (2.22). Their explicit form is obtained using eqs. (A.7), (B.11). We find

$$F_{\rm str+D0} = F_{\rm string} - \frac{VT^{10}2^{11}}{g_{\rm eff}^5}\sum_{w,m=1}^{\infty}\big[1-(-1)^w\big]\big(\frac{m}{w}\big)^5 K_5(2\pi\frac{wm}{g_{\rm eff}})\ . \tag{2.25}$$

3. Hagedorn temperature in string theory

It is useful to recall the way the Hagedorn temperature in superstring theory is manifested in the spectrum, as the temperature at which a certain winding mode becomes massless [5]. We consider the theory in Euclidean space where the time coordinate X^0 is compactified on a circle of circumference $1/T$. In string theory, the presence of coordinates compactified on circles gives rise to winding string states. The string coordinate $X^0(\sigma,\tau)$ can be expanded as follows:

$$X^0(\sigma,\tau) = x^0 + 2\alpha' p^0 \tau + 2R_0 w_0 \sigma + \tilde{X}(\sigma,\tau)\ , \tag{3.1}$$

$$p^0 = \frac{m_0}{R_0}\ , \qquad m_0, w_0 = 0, \pm1, \pm2, ...$$

where $\tilde{X}(\sigma,\tau)$ is a single-valued function of σ and $\int_0^\pi d\sigma \partial_\tau \tilde{X}^0 = 0$. The hamiltonian and level matching constraints are

$$H = \alpha' p_i^2 + \frac{w_0^2 R_0^2}{\alpha'} + \alpha' \frac{m_0^2}{R_0^2} + 2(N_L + N_R - a_L - a_R) = 0\ , \tag{3.2}$$

$$N_L - N_R = m_0 w_0\ .$$

Here a_L, a_R are the normal ordering constants, which represent the vacuum energy of the 1+1 dimensional field theory (e.g. for the bosonic string, $a_L = a_R = 1$). The Hagedorn temperature can be obtained as usual by determining the radius R_0 at which infrared instabilities first appear. We have seen this effect in section 2 in the one-loop contribution to the free energy; in the presence of infrared instabilities, the integral over the torus modular parameter τ_2 diverges at large τ_2. This happens when some state has negative

H, i.e. when a tachyon first appears in the spectrum (apart from the usual bosonic string tachyon). By examining the form of the Hamiltonian, one immediately sees that the first tachyon that appears as the temperature $T = (2\pi R_0)^{-1}$ is increased from zero has $N_L = N_R = 0$, $m_0 = 0$ and $w_0 = \pm 1$. For such states, the critical R_0 is determined by

$$H = 0 = \frac{R_0^2}{\alpha'} - 2(a_L + a_R) \ , \tag{3.3}$$

whereby

$$T_H = \frac{1}{2\pi R_0} = \frac{1}{2\pi\sqrt{2\alpha'(a_L + a_R)}} \ . \tag{3.4}$$

In the NSR formulation of type II superstring theory the calculation is similar. The tachyon appears in the NS-NS sector, where the normal-ordering constants are $a_L = a_R = 1/2$. GSO projection does not remove this tachyon state, because for odd winding number the GSO condition is reversed [1] (this is explicit in the one-loop expression for the free energy in sect. 2; the tachyon state with $w_0 = 0$ is projected out by GSO, but not this thermal tachyon with $w_0 = \pm 1$).

In order to reproduce this calculation in the Green-Schwarz formulation of the superstring (which is more suitable for the generalization to membrane theory), we need to calculate the normal ordering constant for the Euclidean theory on $R^9 \times S^1$. At zero temperature, the normal ordering constant vanishes because of a cancellation between bosons and fermions. In the thermal ensemble at finite temperature, fermions obey antiperiodic boundary conditions under $X^0 \to X^0 + 2\pi R_0$. As a result, supersymmetry is broken and the vacuum energy will not vanish. In type II superstring theory with antiperiodic fermions, the number operators in the sector $w_0 = \pm 1$ are given by

$$N_L = \sum_{n=1}^{\infty} \left[\alpha^i_{-n}\alpha^i_n + (n - \frac{1}{2}) S^a_{-n} S^a_n\right] \ , \quad N_R = \sum_{n=1}^{\infty} \left[\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + (n - \frac{1}{2}) \tilde{S}^a_{-n} \tilde{S}^a_n\right] \ , \tag{3.5}$$

$$i = 1, ..., 8 \ , \qquad a = 1, ..., 8 \ .$$

Thus the normal ordering constant is as in the NS sector of the NSR formulation, i.e. $a_L = a_R = \frac{1}{2}$. In this way we reproduce the result for the Hagedorn temperature in the Green-Schwarz formulation.

The calculation of the normal ordering constants can be done by ζ-function regularization. For the operators in (3.5), one has

$$a_L = a_R = -\frac{1}{2}(D-2)\left[\mathcal{E}_{\rm B} + \mathcal{E}_{\rm F}\right] \ ,$$

with

$$\mathcal{E}_{\rm B} = \sum_{n=1}^{\infty} n \ , \qquad \mathcal{E}_{\rm F} = -\sum_{n=0}^{\infty} (n + \tfrac{1}{2}) \ .$$

Using the formulas

$$\sum_{n=1}^{\infty}\frac{1}{n^\nu}=\zeta(\nu)\ ,\qquad \sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^\nu}=(2^\nu-1)\zeta(\nu)\ ,$$

and $\zeta(-1)=-\frac{1}{12}$, we find

$$\mathcal{E}_{\rm B}=-\frac{1}{12}\ ,\qquad \mathcal{E}_{\rm F}=-\frac{1}{24}\ ,$$

so that for $D=10$ one has $a_L=a_R=\frac{1}{2}$.

4. M-Theory at finite temperature

We will describe M-theory at finite temperature as in sect. 2 by considering the eleven-dimensional theory in euclidean target space with periodic time X^0, and periodic coordinate X^{11}. Fermions are antiperiodic around X^0 and periodic around X^{11}. Thus we are to consider a toroidal compactification of Euclidean M-theory with $(-,+)$ spin structure. Having the topology $R^9\times T^2$, membranes can wrap on a 2-torus X^0,X^{11}.

When viewed in eleven dimensions, the winding string that in sect. 3 led to a tachyon instability is a membrane wrapped around X^0,X^{11} with winding number equal to ±1. Small oscillations of this membrane are effectively described by the $D=11$ supermembrane theory [23]. The observation that there is a tachyon instability at $T>T_{\rm cr}$ will be independent of many details of the membrane Hamiltonian, depending only on the net vacuum energy. Although we will not determine the exact spectrum, being outside of the scope of this paper, it is interesting to note that, because of the supersymmetry breaking boundary conditions, no flat direction remains in the membrane Hamiltonian, so the exact supermembrane spectrum must be discrete (see discussion in sect. 4.2).

In the sector with zero winding, there can be other types of configurations which give rise to low energy excitations, related to D0 brane configurations. We emphasize, however, that the aim here is not to provide a complete account of all relevant excitations of M-theory at a given temperature and radius R_{11}, but rather to point out the existence of a winding mode that becomes tachyonic at some critical temperature $T_{\rm cr}$ (which will depend on the radius of the eleventh dimension). This does not exclude that there could be other instabilities. In particular, in matrix model calculations it has been shown [16] that at some sufficiently high temperature there are D0 branes which cluster at one point. This configuration might lead to a gravitational instability, but estimating the temperature at which such configuration occurs does not appear to be simple [16].

4.1. Toroidal membranes

Before considering the finite temperature case, it is convenient to briefly review the light-cone Hamiltonian formalism for membranes wrapped on a torus in Minkowski space, where X^{10} and X^{11} are compact. Let $\sigma, \rho \in [0, 2\pi)$ be the membrane world-volume coordinates. We can write

$$X^{10}(\sigma,\rho) = w_0 R_{10}\sigma + \tilde{X}^{10}(\sigma,\rho) \ , \qquad X^{11}(\sigma,\rho) = R_{11}\rho + \tilde{X}^{11}(\sigma,\rho) \ , \tag{4.1}$$

where $\tilde{X}^{10}$, $\tilde{X}^{11}$ are single-valued functions of σ and ρ. The transverse coordinates $X^i(\sigma,\rho)$, $i = 1, 2, ..., 8$ are all single-valued (we use the notation where the eleven bosonic coordinates are $\{X^0, X^i, X^{10}, X^{11}\}$). They can be expanded in a complete basis of functions on the torus,

$$X^i(\sigma,\rho) = \sqrt{\alpha'} \sum_{k,m} X^i_{(k,m)} e^{ik\sigma + im\rho} \ , \qquad P^i(\sigma,\rho) = \frac{1}{(2\pi)^2\sqrt{\alpha'}} \sum_{k,m} P^i_{(k,m)} e^{ik\sigma+im\rho} \ ,$$

$$\alpha' = \left(4\pi^2 R_{11} T_2\right)^{-1} \ , \tag{4.2}$$

where T_2 is the membrane tension ($[T_2] = cm^{-3}$). The membrane light-cone Hamiltonian [24,25] takes the form $H = H_0 + H_{\rm int}$, with [26,27]

$$\alpha' H_0 = 8\pi^4 \alpha' T_2^2 R_{10}^2 R_{11}^2 w_0^2 + \frac{1}{2}\sum_{\mathbf{n}} \left[P^i_{\mathbf{n}} P^i_{-\mathbf{n}} + \omega_{km}^2 X^i_{\mathbf{n}} X^i_{-\mathbf{n}}\right]$$

$$\alpha' H_{\rm int} = \frac{1}{4 g_A^2} \sum (\mathbf{n}_1 \times \mathbf{n}_2)(\mathbf{n}_3 \times \mathbf{n}_4) X^i_{\mathbf{n}_1} X^j_{\mathbf{n}_2} X^i_{\mathbf{n}_3} X^j_{\mathbf{n}_4}$$

$$X^+ = \frac{X^0 + \tilde{X}^{11}}{\sqrt{2}} = x^+ + \alpha' p^+ \tau \ ,$$

$$\mathbf{n} \equiv (k, m) \ , \qquad \mathbf{n} \times \ \mathbf{n}' = km' - mk' \ ,$$

$$g_A^2 \equiv \frac{R_{11}^2}{\alpha'} = 4\pi^2 R_{11}^3 T_2 \ , \qquad \omega_{km} = \sqrt{k^2 + w_0^2 m^2 \frac{R_{10}^2}{R_{11}^2}} \ . \tag{4.3}$$

Here only the bosonic modes have been written explicitly (fermion modes will be included later). The constant g_A represents the type IIA string coupling. One can introduce mode operators as follows:

$$X^i_{(k,m)} = \frac{i}{\sqrt{2} w_{(k,m)}} \left[\alpha^i_{(k,m)} + \tilde{\alpha}^i_{(-k,-m)}\right] \ , \qquad P^i_{(k,m)} = \frac{1}{\sqrt{2}} \left[\alpha^i_{(k,m)} - \tilde{\alpha}^i_{(-k,-m)}\right] \ , \tag{4.4}$$

$$\left(X^i_{(k,m)}\right)^\dagger = X^i_{(-k,-m)} \ , \qquad \left(P^i_{(k,m)}\right)^\dagger = P^i_{(-k,-m)} \ , \qquad w_{(k,m)} \equiv \epsilon(k)\, \omega_{km} \ ,$$

where $\epsilon(k)$ is the sign function. The canonical commutation relations imply

$$[X^i_{(k,m)}, P^j_{(k',m')}] = i\delta_{k+k'}\delta_{m+m'}\delta^{ij} \ ,$$

$$[\alpha^i_{(k,m)}, \alpha^j_{(k',m')}] = w_{(k,m)}\delta_{k+k'}\delta_{m+m'}\delta^{ij} \ , \tag{4.5}$$

and similar relations for the $\tilde{\alpha}^i_{(k,m)}$.

The mass operator is given by

$$M^2 = 2p^+p^- - (p^i)^2 - p_{10}^2 = 2H_0 + 2H_{\rm int} - (p^i)^2 - p_{10}^2 \ . \tag{4.6}$$

The Hamiltonian is non-linear. There are two situations where one can extract useful information from this Hamiltonian. One is the limit of large g_A, with R_{10}/R_{11} fixed, in which the non-linear terms are multiplied by the small number $\frac{1}{g_A^2}$ and can be considered in perturbation theory. The other limit is $g_A \to 0$ at fixed R_{10}/R_{11}. This is related to the zero torus area limit of M-theory on T^2, which leads to ten-dimensional type IIB string theory. $H_{\rm int}$ is positive definite, and any state $|\Psi\rangle$ with $\langle\Psi|H_{\rm int}|\Psi\rangle \neq 0$ will have infinite mass in the zero area limit, where $g_A \to 0$ (with $T_2 \to \infty$, so that $\alpha' = (4\pi^2 R_{11} T_2)^{-1}$ and R_{10}/R_{11} remain fixed). The only states that survive are those states made of operators $\alpha^i_{n(p,q)}$, $\tilde{\alpha}^i_{n(p,q)}$ with the same value of (p,q) [27]. They satisfy $\langle\Psi|H_{\rm int}|\Psi\rangle = 0$, so that $H_{\rm int}$ drops out from $\langle\Psi|M^2|\Psi\rangle$. They describe the (p,q) strings of type IIB superstring theory (the proposal that the (p,q) string bound states of type IIB string theory originate from membranes was first made by Schwarz [28]).

Let us now focus on the situation of large g_A. To leading order in perturbation theory in $1/g_A^2$, the interaction term can be dropped. The solution to the membrane equations of motion is given by

$$X^i(\sigma,\rho,\tau) = x^i + \alpha' p^i \tau + i\sqrt{\tfrac{\alpha'}{2}} \sum_{\mathbf{n}\neq(0,0)} w_{\mathbf{n}}^{-1}\left[\alpha^i_{\mathbf{n}} e^{ik\sigma+im\rho} + \tilde{\alpha}^i_{\mathbf{n}} e^{-ik\sigma-im\rho}\right] e^{iw_{\mathbf{n}}\tau} \ . \tag{4.7}$$

Let the momentum components in the directions X^{10} and X^{11} be given by

$$p_{10} = \frac{l_{10}}{R_{10}} \ , \qquad p_{11} = \frac{l_{11}}{R_{11}} \ ,$$

where l_{10}, l_{11} are integers. The nine-dimensional mass operator takes the form $M^2 = \mathcal{H}$, with

$$\mathcal{H} = \frac{l_{10}^2}{R_{10}^2} + \frac{l_{11}^2}{R_{11}^2} + \frac{w_0^2 R_{10}^2}{\alpha'^2} + \frac{1}{\alpha'}\mathbf{H} \ , \tag{4.8}$$

$$\mathbf{H} = \sum_{k,m} \left(\alpha^i_{(-k,-m)}\alpha^i_{(k,m)} + \tilde{\alpha}^i_{(-k,-m)}\tilde{\alpha}^i_{(k,m)}\right) \ . \tag{4.9}$$

The level-matching conditions are given by [29,26]

$$N_\sigma^+ - N_\sigma^- = w_0 l_{10} \ , \qquad N_\rho^+ - N_\rho^- = l_{11} \ , \tag{4.10}$$

where

$$N_\sigma^+ = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{k}{\omega_{km}}\alpha^i_{(-k,-m)}\alpha^i_{(k,m)} \ , \quad N_\sigma^- = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{k}{\omega_{km}}\tilde\alpha^i_{(-k,-m)}\tilde\alpha^i_{(k,m)} \ ,$$

$$N_\rho^+ = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{m}{\omega_{km}}\left[\alpha^i_{(-k,-m)}\alpha^i_{(k,m)} + \tilde\alpha^i_{(-k,m)}\tilde\alpha^i_{(k,-m)}\right] \ ,$$

$$N_\rho^- = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{m}{\omega_{km}}\left[\alpha^i_{(-k,m)}\alpha^i_{(k,-m)} + \tilde\alpha^i_{(-k,-m)}\tilde\alpha^i_{(k,m)}\right] \ .$$

To define the operator $\mathbf{H}$ in the quantum theory, we have to specify the normal ordering prescription. This will give rise to a vacuum energy. The annihilation operators are $\alpha^i_{(k,m)}$, $\tilde\alpha^i_{(k,m)}$ with $k>0$ for all m, and $k=0$, $m>0$. Defining

$$\hat{\mathbf{H}} = \sum_{\mathbf{n}}\left(: \alpha^i_{(-k,-m)}\alpha^i_{(k,m)} : + : \tilde\alpha^i_{(-k,-m)}\tilde\alpha^i_{(k,m)} : \right) \ , \tag{4.11}$$

where the normal ordering symbol "::" means as usual taking the annihilation operators to the right, one finds the relation

$$\mathbf{H} = \hat{\mathbf{H}} + 2(D-3)\mathcal{E} \ , \tag{4.12}$$

$$\mathcal{E} = \frac{1}{2}\sum_{k,m}\omega_{km} \ .$$

This constant shift represents the purely bosonic contribution to the vacuum energy of the 2+1 dimensional field theory (discussed in [30]). If one chooses supersymmetry preserving boundary conditions for fermions, then the fermion and boson contributions to the vacuum energy cancel out [29,23]. Being a consequence of the underlying supersymmetry, this result also holds when non-linear terms are included.

4.2. Vacuum energy for the finite temperature theory

Let us now extend this to the supermembrane theory at finite temperature. The euclidean time coordinate X^0 plays role of X^{10}. Fermions will obey antiperiodic boundary conditions around X^0, and periodic boundary conditions around X^{11}. We are interested in the sector $w_0 = \pm 1$, where fermions are antiperiodic under $\sigma \to \sigma + 2\pi$. This implies that the frequencies k in the Fourier expansions will be half integers, and the frequencies m

will be integers (since fermions are periodic under $\rho \to \rho + 2\pi$). The Hamiltonian operator is (cf. (3.5))

$$\mathcal{H} = \frac{l_0^2}{R_0^2} + \frac{l_{11}^2}{R_{11}^2} + \frac{R_0^2}{\alpha'^2} \ + \frac{1}{\alpha'}(\hat{\mathbf{H}} + 2(D-3)\mathcal{E}) \ , \tag{4.13}$$

where

$$\hat{\mathbf{H}} = \sum_{\mathbf{n}} \big[: \alpha^i_{-\mathbf{n}} \alpha^i_{\mathbf{n}} : + : \tilde{\alpha}^i_{-\mathbf{n}} \tilde{\alpha}^i_{\mathbf{n}} : + \omega_{k+\frac{1}{2},m} \big(: S^a_{-\mathbf{n}} S^a_{\mathbf{n}} : + : \tilde{S}^a_{-\mathbf{n}} \tilde{S}^a_{\mathbf{n}} : \big) \big] \ ,$$

and

$$\mathcal{E} = \mathcal{E}_{\rm B} + \mathcal{E}_{\rm F} = \frac{1}{2} \sum_{k,m} \big(\omega_{km} - \omega_{k+\frac{1}{2},m} \big) \ , \tag{4.14}$$

$$\omega_{km} = \left(k^2 + \frac{m^2}{g_{\rm eff}^2} \right)^{\frac{1}{2}} \ . \tag{4.15}$$

The sums in (4.14) are divergent, but they can be defined by analytic continuation. The procedure generalizes the usual zeta-function regularization used in sect. 3 for the superstring case, and it is equivalent to the functional relation $E_\nu(\Omega) = cE_{1-\nu}(\Omega)$ allowing the definition of Eisenstein series with $\nu < 1/2$ [17]. We write

$$\begin{aligned} \mathcal{E} &= \lim_{\nu \to -\frac{1}{2}} \ \frac{1}{2} \sum_{k,m} \left(\frac{1}{(\omega_{km})^{2\nu}} - \frac{1}{(\omega_{k+\frac{1}{2},m})^{2\nu}} \right) \\ &= \frac{\pi^\nu}{2\Gamma(\nu)} \sum_{k,m} \int_0^\infty \frac{d\tau}{\tau} \tau^\nu \left(e^{-\pi\tau\omega_{km}^2} - e^{-\pi\tau(\omega_{k+\frac{1}{2},m})^2} \right) \ . \end{aligned} \tag{4.16}$$

Then, using the Poisson formula (2.7) one obtains

$$\mathcal{E} = \frac{\pi^\nu g_{\rm eff}}{2\Gamma(\nu)} \sum_{w,w} \int_0^\infty \frac{ds}{s} \ s^{1-\nu} \big(1 - (-1)^w \big) \exp \big[-\pi s (w^2 + w'^2 g_{\rm eff}^2) \big] \ , \tag{4.17}$$

where we have made the change of integration variable, $s = 1/\tau$. Hence

$$\mathcal{E} = \frac{g_{\rm eff} \pi^{2\nu-1} \Gamma(1-\nu)}{2\Gamma(\nu)} \sum_{w,w'} \big(1 - (-1)^w \big) \frac{1}{(w^2 + w'^2 g_{\rm eff}^2)^{1-\nu}} \ . \tag{4.18}$$

Setting now $\nu = -1/2$ we get

$$\mathcal{E} = -\frac{g_{\rm eff}}{8\pi^2} \sum_{(w,w') \neq (0,0)} \big(1 - (-1)^w \big) (w^2 + w'^2 g_{\rm eff}^2)^{-\frac{3}{2}} \ . \tag{4.19}$$

Thus we have (see eqs. (A.1), (B.1))

$$\mathcal{E} = -\frac{1}{4\pi^2 \sqrt{g_{\rm eff}}} \ \zeta(3) \big(E_{\frac{3}{2}}(g_{\rm eff}) - F_{\frac{3}{2}}(g_{\rm eff}) \big) \ . \tag{4.20}$$

Using the expansions (A.5), (A.6), (B.9), (B.10) the vacuum energy takes the form

$$\mathcal{E} = -\frac{1}{8} + O(e^{-2\pi/g_{\rm eff}}) \ , \qquad g_{\rm eff} \ll 1 \ , \tag{4.21}$$

and

$$\mathcal{E} = -\frac{7}{16\pi^2}\zeta(3) g_{\rm eff} - \frac{1}{12 g_{\rm eff}} + O(e^{-2\pi g_{\rm eff}}) \ , \qquad g_{\rm eff} \gg 1 \ . \tag{4.22}$$

The explicit analytic form for the exponentially small terms can be read from the formulas in the appendices.

Notably, eq. (4.21) implies that at $g_{\rm eff} \ll 1$ (i.e. small type IIA coupling g_A or sufficiently low temperatures), the vacuum energy is identical to that of type II superstring theory, i.e. $2(D-2)\mathcal{E} = -2(D-2)(\frac{1}{12} + \frac{1}{24})$, due to a cancellation of the term proportional to $\zeta(3)/g_{\rm eff}^{3/2}$ in the expansions (A.5), (B.9).

A question is why the vacuum energy gives the correct result in the weak coupling limit $g_A \ll 1$. In the derivation of the vacuum energy, we have used the assumption that $g_A \gg 1$ to neglect the contribution of the non-linear terms. The fact that the correct result emerges at weak coupling indicates that a possible extra contribution coming from the non-linear terms in the Hamiltonian may cancel out between fermion and boson contributions.

Another interesting point is the issue of flat directions in the membrane Hamiltonian for the wrapped membrane. Consider first the case of supersymmetric boundary conditions. In the strict limit $g_A \to \infty$, one has a Hamiltonian which is a sum of harmonic oscillators, so there is no flat direction and the spectrum is discrete. For any finite $g_A \gg 1$, states representing small oscillations should be almost stable, since they only see the harmonic potential. However, if flat directions are present, they may decay by tunnel effect (see also discussion in [31]). This effect should be exponentially small for large g_A. Now, in the present case of non-supersymmetric boundary conditions, possible flat directions will be removed by the same effect flat directions are removed in the bosonic theory (described in [32]). The motion is confined to some finite region, and the exact spectrum of the supermembrane must be discrete. For large values of g_A, most of the states are confined to the harmonic region of the potential, so this effect should not play a significant role.

4.3. Critical temperature in M Theory

As in the string theory case, there will be a tachyonic instability when $\mathcal{H} < 0$ for some state (see (4.13)). The first state that solves $\mathcal{H} = 0$ is a state with $l_0 = l_{11} = 0$, $w_0 = \pm 1$, which is annihilated by all annihilation operators $\alpha^i_{\mathbf{n}}$, $\tilde{\alpha}^i_{\mathbf{n}}$ (this is nothing but the "uplift" of the winding tachyon of string theory to eleven dimensions). From eq. (4.13), we thus find that the critical temperature is determined by the solution of the equation

$$\mathcal{H} = 0 = \frac{R_0^2}{\alpha'} + 2(D-3)\mathcal{E} \ . \tag{4.23}$$

Using eq. (4.20) and setting $D = 11$, this becomes

$$\frac{1}{T_{\rm cr}^2} = \alpha' \frac{16}{\sqrt{g_{\rm eff}}}\ \zeta(3)\big(E_{\frac{3}{2}}(g_{\rm eff}) - F_{\frac{3}{2}}(g_{\rm eff})\big)\ . \tag{4.24}$$

Here $g_{\rm eff} = 2\pi\sqrt{\alpha'}T_{\rm cr}g_A$, so this is a transcendental equation for $T_{\rm cr}$. It can be solved analytically in two regimes, $g_{\rm eff} \ll 1$ and $g_{\rm eff} \gg 1$, using the expansions (4.21) and (4.22). At weak coupling $g_{\rm eff} \ll 1$, the equation (4.23) becomes

$$\frac{R_0^2}{\alpha'} = -16\ \left(-\frac{1}{8} + O(e^{-\frac{2\pi}{g_{\rm eff}}})\right)\ , \tag{4.25}$$

i.e.

$$T_{\rm cr} = \frac{1}{2\pi\sqrt{2\alpha'}}\ . \tag{4.26}$$

This coincides with the Hagedorn temperature. This is a consequence of the observation of the previous subsection that the vacuum energy reduces to the weak coupling superstring value at $g_{\rm eff} \ll 1$. Since the critical temperature in this regime is of order $1/\sqrt{\alpha'}$, the condition $g_{\rm eff} \ll 1$ implies $g_A \ll 1$.

It is easy to see that in a regime $g_A \gg 1$, the coupling $g_{\rm eff}$ will be large at the critical temperature $T = T_{\rm cr}$. This means that, in order to determine the critical temperature, we have to use (4.22). Using (4.22) and keeping only the leading term, the condition (4.24) determining the critical temperature becomes

$$\frac{1}{T_{\rm cr}^2} \cong \alpha' 28\zeta(3)g_{\rm eff} = 56\pi{\alpha'}^{3/2}\zeta(3)g_A\ T_{\rm cr}\ . \tag{4.27}$$

Thus the critical temperature at strong coupling is

$$T_{\rm cr} \cong \frac{1}{a\sqrt{\alpha'}(2\pi g_A)^{\frac{1}{3}}} = \frac{1}{a}\big(2\pi T_2\big)^{1/3}\ , \qquad g_A \gg 1\ , \tag{4.28}$$

$$a = \big[28\zeta(3)\big]^{\frac{1}{3}} \cong 3.23\ ,$$

where T_2 is the membrane tension (we have used eqs. (4.2), (4.3)). In terms of the eleven-dimensional Planck length l_P, $g_A^2 = 2\pi\frac{R_{11}^3}{l_P^3}$, so

$$T_{\rm cr} = \frac{1}{a\ l_P} \cong 0.31\ l_P^{-1}\ \ . \tag{4.29}$$

Here l_P is normalized so that the gravitational coupling is $\kappa_{11}^2 = 16\pi^5 l_P^9$. As a check, note that $g_{\rm eff}(T_{\rm cr}) = 2\pi\sqrt{\alpha'}T_{\rm cr}g_A \sim g_A^{2/3} \gg 1$, which is consistent with approximating the Eisenstein functions by eq. (4.22). The same result (4.28) is obtained by solving (4.24) numerically at $g_A \gg 1$. Thus the critical temperature for the type IIA superstring

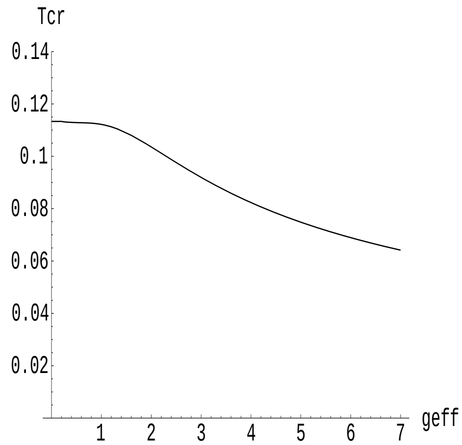

Fig. 1: Critical temperature as a function of $g_{\rm eff}$ (with $\alpha' = 1$). The value at $g_{\rm eff} = 0$ is $T_H = 1/(2\pi\sqrt{2\alpha'})$.

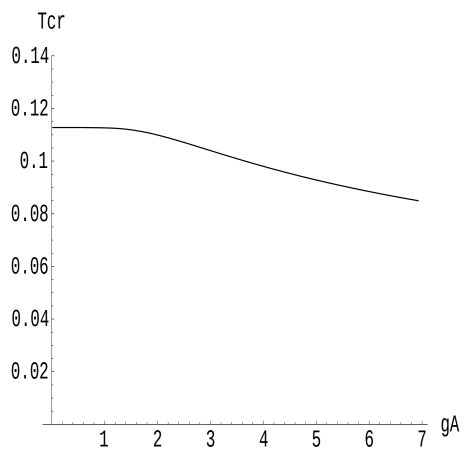

Fig. 2: Critical temperature as a function of g_A.

decreases at strong coupling. In terms of the eleven-dimensional Planck length, at large radius, it approaches a constant value, $T_{\rm cr} \cong 0.31\ l_P^{-1}$.

By studying an expression for the free energy computed in a semiclassical approximation, in ref. [11] a "regularized" Hagedorn temperature was proposed for the $D = 11$ theory, which becomes infinity as the cutoff is sent to zero. The numerical coefficient contains a similar factor $7\zeta(3)$ as in (4.27). The appearance of this factor in [11] is also related to the vacuum energy of the world-volume theory in $D = 11$.

The critical temperature can be obtained for all values of the coupling by solving eq. (4.24) numerically. Fig. 1 is a plot of the critical temperature as a function of $g_{\rm eff}$, and fig. 2 is a plot of $T_{\rm cr}$ as a function of g_A. At small couplings, the plots have the same behavior, since $T_{\rm cr}$ is approximately constant. At strong coupling, $T_{\rm cr}$ goes to zero as $1/g_{\rm eff}^{1/2}$ in fig. 1, and as $1/g_A^{1/3}$ in fig. 2.

The natural mass scale in eleven dimensions is l_P^{-1}. This means that the energies of elementary excitations of the system at large g_A must be proportional to l_P^{-1}, not $1/\sqrt{\alpha'}$. So temperature at $g_A \gg 1$ is more properly measured in units of l_P^{-1}. An analogous situation happens in type IIB superstring theory. The Hagedorn temperature at $g_B \gg 1$ can be found by S-duality. The strong coupling limit of type IIB theory is known to be the same theory, where fundamental strings are replaced by D-strings, $g_B \to 1/g_B$ and α' by $\alpha'_D = g_B\alpha'$, so that the Hagedorn termperature is

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'_D}} = \frac{1}{2\pi\sqrt{2\alpha' g_B}} \ .$$

The Hagedorn temperature goes to zero for large g_B at fixed α', but it should be measured with respect to the D-string tension (since elementary excitations have energies of order $1/\sqrt{\alpha'_D}$), in which case it is a constant independent of the coupling. In principle, it seems possible to generalize the present method to determine a critical temperature in type IIB superstring theory at some intermediate coupling g_B. In order to connect M-theory at finite temperature with type IIB theory one needs to study M-theory on an euclidean 3-torus. Now membranes can wrap in different ways.

4.4. Duality connections

The Hagedorn temperature in string theory was first understood as a consequence of the exponential growth of the asymptotic level density with the mass, $\rho(m) \sim e^{\mathrm{const}.m}$. The existence of a finite critical temperature at $g_A \gg 1$ can be explained if there is a string-theoretic description of M-theory in this limit. The tension $1/\tilde{\alpha}'$ of such string can be read (up to a numerical constant) from the critical temperature. We have the formula:

$$T_{\rm cr}^2 = \frac{1}{28\zeta(3)\alpha' g_{\rm eff}} = \frac{\rm const.}{\tilde{\alpha}'} \ . \tag{4.30}$$

How can a string-theory description arise at large R_{11} ? In this limit, near the critical temperature we have $R_0 \ll R_{11}$. Therefore, the relevant low energy degrees of freedom of the system are more appropriately described by making dimensional reduction along the Euclidean time direction X^0, with X^{11} now playing the role of a compact spatial dimension of the resulting ten-dimensional theory. Because of the antiperiodic boundary conditions around X^0, the resulting ten dimensional string theory is a non-supersymmetric string theory. According to [21], a compactification of M-theory on a circle where fermions obey antiperiodic boundary conditions gives type 0A string theory (the relation between finite temperature type IIA theory and type 0A theory was noted already in [1]). Therefore the strong coupling limit of type IIA superstring theory at finite temperature T would be described by euclidean type 0A string theory, where the string coupling is $\tilde{g}_A^2 = 2\pi\frac{R_0^3}{l_P^3} =$

$(4\pi^2 T^3 l_P^3)^{-1}$, and the string tension is obtained from the usual formula $\tilde{\alpha}' = \frac{l_P^3}{2\pi R_0} = l_P^3 T = \alpha' g_{\rm eff}$. This agrees with the identification in (4.30). In the strict limit $R_{11} \to \infty$, this duality implies that *uncompactified* M-theory at temperature T is described by a ten-dimensional euclidean string theory.

Reproducing the numerical coefficient in (4.30) using string theory techniques may not be simple, because the type 0A theory is strongly coupled below the critical temperature, i.e. $\tilde{g}_A > O(1)$ for $T < T_{\rm cr}$. However, eq. (4.30) predicts that the type 0A tachyon must disappear at a coupling

$$\tilde{g}_A^2 > \tilde{g}_{A\rm cr}^2 = \frac{7\zeta(3)}{\pi^2} \cong 0.85 \ .$$

The precise numerical value may be subject to corrections, for reasons explained in sect. 1. This agrees with the suggestion of [21], that the type 0A tachyon should become massive at strong coupling. Conversely, the existence of a critical coupling $\tilde{g}_{A\rm cr}$ in type 0A string theory at which the type 0A tachyon becomes massless implies the existence of the critical temperature $T = O(l_P^{-1})$ found in this paper in uncompactified M-theory at finite temperature. In terms of the critical coupling, the critical temperature is $T_{\rm cr} = (2\pi g_{A\rm cr})^{-2/3}\ l_P^{-1}$. It would be interesting to investigate further consequences of this connection in more detail.

Acknowledgements

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Appendix A. Non-holomorphic Eisenstein series

The non-holomorphic Eisenstein series is defined by [17]

$$2\zeta(2r)E_r(\Omega) = \sum_{(k,m)\neq(0,0)} \frac{\Omega_2^r}{|k+m\Omega|^{2r}} \ , \qquad r > \frac{1}{2} \ , \tag{A.1}$$

where $\Omega = \Omega_1 + i\Omega_2$ is a complex parameter describing the upper half complex plane. The Eisenstein series $E_r(\Omega)$ is invariant under $SL(2,Z)$ transformations of Ω,

$$\Omega \to \frac{a\Omega + b}{c\Omega + d} \ , \quad ad - bc = 1 \ , \quad a,b,c,d \ \in \ Z \ . \tag{A.2}$$

At large Ω_2, one has the expansion

$$E_r(\Omega) = \Omega_2^r + \gamma_r \Omega_2^{1-r} + \frac{4\Omega_2^{1/2}\pi^r}{\zeta(2r)\Gamma(r)} \sum_{n,w=1}^{\infty} \big(\frac{w}{n}\big)^{r-1/2} \cos(2\pi w n\Omega_1) K_{r-1/2}(2\pi w n \Omega_2) \ , \tag{A.3}$$

$$\gamma_r = \frac{\sqrt{\pi}\ \Gamma(r-1/2)\ \zeta(2r-1)}{\Gamma(r)\ \zeta(2r)}\ .$$

The derivation is as in the analogous case given in appendix B. Using the asymptotic expansion for the Bessel function $K_{r-1/2}$,

$$K_{r-1/2}(2\pi w n\Omega_2) = \frac{1}{\sqrt{4wn\Omega_2}} e^{-2\pi w n\Omega_2} \sum_{m=0}^{\infty} \frac{1}{(4\pi w n\Omega_2)^m} \frac{\Gamma(r+m)}{\Gamma(r-m)m!}\ , \tag{A.4}$$

we see that the terms in (A.3) involving the Bessel function will be exponentially suppressed. In the present case with $\Omega_2 = \frac{1}{g_{\rm eff}}$, such exponentially suppressed terms represent non-perturbative contributions originating from $D0$ branes, whose coefficient is therefore exactly determined by the above expansion of the Bessel function. In a strong coupling expansion – obtained by the modular transformation $\Omega \to -\Omega^{-1}$ – the exponentially suppressed terms are instead of the form $e^{-2\pi g_{\rm eff}}$.

From equation (A.3), we obtain the following expressions for the expansions of Eisenstein series appearing in sects. 2 and 4:

$$\zeta(3)E_{\frac{3}{2}}(g_{\rm eff}) = \frac{\zeta(3)}{g_{\rm eff}^{\frac{3}{2}}} + \frac{\pi^2}{3}\ g_{\rm eff}^{\frac{1}{2}} + \frac{8\pi}{\sqrt{g_{\rm eff}}} \sum_{n,w=1}^{\infty} \frac{w}{n} K_1(2\pi \frac{wn}{g_{\rm eff}})\ , \quad \text{for}\quad g_{\rm eff} \ll 1\ , \tag{A.5}$$

$$\zeta(3)E_{\frac{3}{2}}(g_{\rm eff}) = \zeta(3) g_{\rm eff}^{\frac{3}{2}} + \frac{\pi^2}{3}\ \frac{1}{g_{\rm eff}^{\frac{1}{2}}} + 8\pi\sqrt{g_{\rm eff}} \sum_{n,w=1}^{\infty} \frac{w}{n} K_1(2\pi w n g_{\rm eff})\ , \quad \text{for}\quad g_{\rm eff} \gg 1\ , \tag{A.6}$$

and

$$\begin{aligned} \zeta(11)E_{\frac{11}{2}}(g_{\rm eff}) &= \zeta(11) g_{\rm eff}^{-11/2} + \frac{256\zeta(10)}{315} g_{\rm eff}^{9/2} \\ &+ \frac{4\pi^{11/2}}{\Gamma(\frac{11}{2})\sqrt{g_{\rm eff}}} \sum_{n,w=1}^{\infty} \big(\frac{w}{n}\big)^5 K_5(2\pi\frac{wn}{g_{\rm eff}})\ , \qquad g_{\rm eff} \ll 1\ , \end{aligned} \tag{A.7}$$

$$\begin{aligned} \zeta(11)E_{\frac{11}{2}}(g_{\rm eff}) &= \zeta(11) g_{\rm eff}^{11/2} + \frac{256\zeta(10)}{315} g_{\rm eff}^{-9/2} \\ &+ \frac{4\pi^{11/2}\sqrt{g_{\rm eff}}}{\Gamma(\frac{11}{2})} \sum_{n,w=1}^{\infty} \big(\frac{w}{n}\big)^5 K_5(2\pi w n g_{\rm eff})\ , \qquad g_{\rm eff} \gg 1\ . \end{aligned} \tag{A.8}$$

Appendix B. Generalized Eisenstein series for fermion contributions

In the calculations performed in the main text, fermion contributions (either to the free energy or to the vacuum energy) led to Eisenstein-type series of the form

$$2\zeta(2r)F_r(\Omega) \equiv \sum_{(k,m)\neq(0,0)} (-1)^m \frac{\Omega_2^r}{|k+m\Omega|^{2r}}\ , \quad r > \tfrac{1}{2}\ . \tag{B.1}$$

Here we will derive some basic properties that we need, such as the weak and strong coupling expansions.

Note that $F_r(\Omega)$ is *not* $SL(2,Z)$ invariant, so the weak and strong coupling expansions will be different. In particular, the modular transformation $\Omega \to -1/\Omega$ gives

$$2\zeta(2r)F_r(-1/\Omega) \equiv \sum_{(k,m)\neq(0,0)} (-1)^k \frac{\Omega_2^r}{|k+m\Omega|^{2r}} \ . \tag{B.2}$$

Let us first derive an expansion of (B.1) for $\Omega_2 \gg 1$. We will make use of the formulas:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \ , \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -\zeta(s)(1-2^{1-s}) \ .$$

Separating the $m=0$ term in eq. (B.1), we get

$$\zeta(2r)F_r(\Omega) = \zeta(2r)\Omega_2^r + \Omega_2^r \sum_k \sum_{m=1}^{\infty} (-1)^m \frac{\pi^r}{\Gamma(r)} \int_0^{\infty} \frac{dx}{x} x^r e^{-\pi x|k+m\Omega|^2} \ . \tag{B.3}$$

We now use the Poisson resummation formula,

$$\sum_k f(k) = \sum_{k'} \int_{-\infty}^{\infty} d\mu \ f(\mu) \ e^{2\pi i\mu k'} \ . \tag{B.4}$$

We get

$$\zeta(2r)F_r(\Omega) = \zeta(2r)\Omega_2^r + \frac{\Omega_2^r \pi^r}{\Gamma(r)} \sum_{k'} \sum_{m=1}^{\infty} (-1)^m e^{2\pi i k' m\Omega_1} \int_0^{\infty} \frac{dx}{x} x^{r-\frac{1}{2}} e^{-\pi x m^2\Omega_2^2 - \frac{\pi k'^2}{x}} \tag{B.5}$$

Separating the term $k'=0$, and performing the integrations, we finally obtain

$$\begin{aligned} \zeta(2r)F_r(\Omega) &= \zeta(2r)\Omega_2^r + \beta_r \Omega_2^{1-r} \\ &+ \frac{4\Omega_2^{1/2}\pi^r}{\Gamma(r)} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} (-1)^m \Big(\frac{k}{m}\Big)^{r-1/2} \cos(2\pi k m\Omega_1) K_{r-1/2}(2\pi k m \Omega_2) \ , \end{aligned} \tag{B.6}$$

$$\beta_r \equiv -\frac{\sqrt{\pi}\Gamma(r-1/2)}{\Gamma(r)} \zeta(2r-1) \left(1-2^{2-2r}\right) \ .$$

For $\Omega = i/g_{\rm eff}$, eq. (B.6) is an expansion which is applicable in the regime $g_{\rm eff} \ll 1$. An expansion for the opposite regime, $g_{\rm eff} \gg 1$, can be obtained by proceeding in a similar way, but separating the $k=0$ term in eq. (B.1). Define $\tilde\Omega = -1/\Omega$, so that $\tilde\Omega_2 = \frac{\Omega_2}{|\Omega|^2}$, $\tilde\Omega_1 = -\frac{\Omega_1}{|\Omega|^2}$. We get

$$\zeta(2r)F_r(\Omega) = -\zeta(2r)(1-2^{1-2r})\tilde\Omega_2^r + \tilde\Omega_2^r \sum_m \sum_{k=1}^{\infty} (-1)^m \frac{\pi^r}{\Gamma(r)} \int_0^{\infty} \frac{dx}{x} x^r e^{-\pi x|m+k\tilde\Omega|^2} \ . \tag{B.7}$$

Now we make Poisson resummation in m, and then perform the integration over x. We obtain

$$\begin{aligned}\zeta(2r)F_r(\Omega) &= -\zeta(2r)(1-2^{1-2r})\tilde{\Omega}_2^r \\ &\quad + \frac{2\tilde{\Omega}_2^{1/2}\pi^r}{\Gamma(r)}\sum_m\sum_{k=1}^{\infty}\Big(\frac{|m+\frac{1}{2}|}{k}\Big)^{r-\frac{1}{2}} e^{2\pi ik(m+\frac{1}{2})\tilde{\Omega}_1} K_{r-\frac{1}{2}}(2\pi k|m+\tfrac{1}{2}|\tilde{\Omega}_2)\ .\end{aligned} \tag{B.8}$$

This expansion is applicable for large $\tilde{\Omega}_2$. In our case, we have $\Omega = i/g_{\rm eff}$ and $\tilde{\Omega} = ig_{\rm eff}$, so eq. (B.8) gives an expansion for $g_{\rm eff} \gg 1$. Note that there is only one power-like term, and the remaining terms are exponentially suppressed at large $\tilde{\Omega}_2$.

Summarizing, we obtain for $r = 3/2$ and $r = 11/2$ the following expansions (cf. (A.5)–(A.8))

$$\zeta(3)F_{\frac{3}{2}}(g_{\rm eff}) = \frac{\zeta(3)}{g_{\rm eff}^{\frac{3}{2}}} - \frac{\pi^2}{6}g_{\rm eff}^{\frac{1}{2}} + \frac{8\pi}{\sqrt{g_{\rm eff}}}\sum_{k,m=1}^{\infty}(-1)^m\frac{k}{m}K_1(2\pi\frac{km}{g_{\rm eff}})\ , \qquad g_{\rm eff} \ll 1\ , \tag{B.9}$$

$$\zeta(3)F_{\frac{3}{2}}(g_{\rm eff}) = -\frac{3}{4}\zeta(3)g_{\rm eff}^{\frac{3}{2}} + 4\pi\sqrt{g_{\rm eff}}\sum_m\sum_{k=1}^{\infty}\frac{|m+\frac{1}{2}|}{k}K_1(2\pi g_{\rm eff}k|m+\tfrac{1}{2}|)\ ,\quad g_{\rm eff} \gg 1, \tag{B.10}$$

and

$$\begin{aligned}\zeta(11)F_{\frac{11}{2}}(g_{\rm eff}) &= \frac{\zeta(11)}{g_{\rm eff}^{11/2}} - \frac{256}{315}(1-2^{-9})\zeta(10)g_{\rm eff}^{9/2} \\ &\quad + \frac{4\pi^{11/2}}{\Gamma(\frac{11}{2})\sqrt{g_{\rm eff}}}\sum_{k,m=1}^{\infty}(-1)^m\Big(\frac{k}{m}\Big)^5 K_5(2\pi\frac{km}{g_{\rm eff}})\ , \qquad g_{\rm eff} \ll 1\ ,\end{aligned} \tag{B.11}$$

$$\begin{aligned}\zeta(11)F_{\frac{11}{2}}(g_{\rm eff}) &= -(1-2^{-10})\zeta(11)g_{\rm eff}^{11/2} \\ &\quad + \frac{2\pi^{11/2}\sqrt{g_{\rm eff}}}{\Gamma(\frac{11}{2})}\sum_m\sum_{k=1}^{\infty}\frac{|m+\frac{1}{2}|^5}{k^5}K_5(2\pi g_{\rm eff}k|m+\tfrac{1}{2}|)\ , \qquad g_{\rm eff} \gg 1\ .\end{aligned} \tag{B.12}$$

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